Abstract

In this paper, we propose a methodology for documentation and geometrical analysis of vaults that relies on laser scanning, geometric analysis, three-dimensional modelling, and 3D printing. The methodology was applied to the study of three types of vaults, two of them in brick and the other, the main vault of the Arco da Rua Augusta, in stone stereotomy. Starting from a planimetric layout a geometric analysis is carried out to infer the generative geometries. This is done by exploring the point cloud data and other related outputs, like drawings or meshes. A return to three-dimensionality was achieved through digital modelling and 3D printing. Returning to physicality allows a different type of interaction where some aspects of constructive systems, like assembly hypothesis, can be tested. The proposed methodology paves the way to recognize heritage significance from direct documentation which is fundamental to support conservation studies.

keywords

VERNACULAR ARCHITECTURE
BRICK VAULT
STEREOTOMY
DESCRIPTIVE GEOMETRY
LASER SCANNING
3D PRINTING
ARCHITECTURAL HERITAGE VALUES

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From the empirical making of Alentejo vaults
to the generative geometry of the Arco da Rua Augusta vault

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Introduction

The main purpose of this paper is to show that surveying methods like photogrammetry and laser scanning, while ensuring precision and quality of geometric data acquisition allow the development of rigorous and quantitative studies which pave the way for more informed interventions in heritage. To demonstrate this statement, we present a methodology to analyse, from a geometric point of view, the vault constructive element. The methodology includes the following steps: 1) laser scanning survey and mesh modelling, 2) geometric analysis and modelling, 3) analysis of deviations between the reality of the constructed fabric, given by the surveyed data, and the idealised geometric model, and 4) 3D printing.

To apply the methodology, we have chosen three situations of vault construction that pose different challenges due to its nature an erudition level. These range from a type of vault that is constructed mainly based in empirical rules in a context of vernacular architecture, to a stereotomic vault based in a very sophisticated design process. By applying the methodology to these different types of vaults we will be able to understand different levels of depth in the use of geometry as a conceptual tool to generate the built form.

The remaining of the paper is structured as follows. In the next two sections we do a short review of the literature and an introduction to the case studies. Then, the following three sections address the application of the analysis methodology to each of the case studies. Finally, in the last section we discuss the results and present the conclusions.

A short review of the constructive element vault and related works

A vault is a construction solution for covering a span framed by supports that can be in the form of walls or columns. It generally has a curved shape and can be simple, consisting of a single surface, or composite, consisting of multiple surfaces that connect with each other (Rodrigues et al. 2002; Escudero et al. 2014). It features a visible surface, the intrados, and a generally invisible surface, the extrados. There are exceptions to this rule, for example, in the case of some domes, which are special cases of vaults where the extrados is a visible surface. It is a construction system that, like the arch, is based on the balance generated by compressive forces resulting from its own weight and any additional loads placed on the extrados, or others, such as the bells suspended in bell towers that help balance the vault that covers them. It is present in many types of architecture, both vernacular and erudite. It can be constructed with various materials such as earth, brick masonry, and stone, and its design can be based on empirical rules, refined over years of practice and tradition, as is the case with medieval and Gothic vaults, traditional Alentejo vaults, or some types of brick vaults commonly seen in civil architecture, or it may involve a prior design, drafted, which became more common starting from the Renaissance, particularly in stone stereotomies (González 2006; Calvo-López 2020).

Stone stereotomy is the study of cutting this material for construction purposes, namely for arches, domes and vaults, primarily with a structural purpose, based on geometric rules. This construction tradition in Western Europe lasted until the end of the 19th century and tended to disappear in the beginning of the 20th century due to the use of reinforced concrete. This knowledge is preserved through architectural treatises like those of Derand (1643) or Frézier (1739), just to mention a few. On the other hand, it awaits discovery through the analysis of built structures, which are, in fact, the place placeholder of that knowledge. Recently, there have been several studies regarding stereotomy. They typically cover specific geometric types, periods, or regions (Callabria 2015; Calvo-López 2020; Delgado 2017; Díaz 1996, 2020; Genin et al. 2009; Genin 2014, 2020; Huerta 2007; Palacios 1990; Pérouse de Montclos 2001; Sakarovitch 1997; Shelby 1969). Montclos (2001) mainly studied the French case, with some forays into Spain, and briefly into Italy, Germany, and England. Regarding the Spanish case, Rabasa Díaz (2000) studied ribbed vaults from the Renaissance to the 19th century, and Palacios (1990) conducted research on the influence of Vandelvira's treatise, from 1585, on the Spanish Renaissance, where he reinterprets drawings and shows constructed examples.

The first reference to Portugal appears in Frézier's treatise, *La théorie et la pratique de coupe des pierres* (...) (1737-1739). In this work, the vaults of the Church of the Jerónimos Monastery in Belém are described as 'the most beautiful and well-executed of their kind' (Frezier 1739, vol. 3, 28). Genin (2014, 2020) and Delgado (2017), mentioned by Calvo-López (2020) and Calabria (2015), investigated the stereotomy of the 16th century. Sousa (1988) studied some examples of skew arches in tunnels and bridges in the Lisbon region. Regarding the period between the 18th century and the end of the 19th century in Portugal, there are no systematic studies. Exemplars that deserve study from the perspective of stone stereotomy include the National Palace of Mafra, the Estrela Basilica, the Church of Memory, the Galo Tower, and the Arco da Rua Augusta, all located in the Lisbon region or its vicinity.

To perform a geometric study on this, or other, historic constructive element, one must rely on a survey. The surveys can be done using a wide variety of methods and tools. These can include direct measurements, topographic surveying, photogrammetry and laser scanning. In the context of this paper, we are particularly interested in demonstrating that laser scanning as a data collection tool provides a very useful support for precise quantitative studies. The works from Genin (2009, 2014, 2020) relied on photogrammetry as the documentation method. Mateus et al. (2012) and Calabria (2015) mention the use of laser scanning as the primary method for data acquisition. Many studies report on the use of laser scanning and/or photogrammetry as the primary methods for data acquisition to support geometrical analysis of built fabric.

The motivation for such studies is varied. Banfi (2019) and Stanga et al. (2019) describe methodologies for vault modelling in the context of BIM that include in situ observation, point cloud survey, and modelling. However, the focus is placed in fitting NURBS primitives to point cloud data, to get the smaller possible deviations, rather that understanding the geometric nature of the geometries involved. Other studies are more focused on understanding and interpreting the geometric nature of the constructive elements. Capone et al. (2019), Spallone et al. (2019) and Vitali et al. (2021) developed studies about vaulted systems based on archival drawings, treatises and photogrammetric and laser scanning surveys. Scandurra et al. (2018) discuss the geometrical nature of the dome of the church of San Carlo all'Arena in Naples based in a laser scanning and photogrammetric survey. Essentially the authors experimented between oval and elliptic solutions for the planimetric shape of the dome and then used a three-axis ellipsoid as basis for the 3D model. The end purpose was to generate a BIM model of the church. In the process a reference to Serlio treatise is made, which highlights the possible importance of historical data. López González et al. (2023) present a study about the Valencia Cathedral in which, after a laser scanning is carried out, a planimetric and sectional analysis is carried out. An interesting aspect of this analysis is to consider the ancient units of measurement. By doing so, the system of proportions becomes much more apparent.

Another category of studies, like the ones developed by Agustín-Hernández et al. (2021) and López González et al. (2020) consider and underline the role of point clouds as the primary data for the geometrical interpretation and analysis. In this line, Lanzara et al. (2019) define a workflow aiming reliable reconstruction hypothesis of vaulted systems including point cloud survey, geometric modelling, considering both ideal models and models mathematically fit to the surveyed data, and comparison between the models and the survey data. Rinaudo et al. (2023) present a methodology for the interpretation of point clouds through the geometric study of key sections, leading to imposition of some theoretical models to the surveyed the data. Then these models are compared with the survey in terms of distances. The methodology includes the production of haptic models via laser cutting and 3D printing. An interesting aspect of this study is to consider geometry as a value qualifying intangible heritage. Finally, Attenni et al. (2023) present a methodology to analyse vaulted systems very similar to the one we present in this paper except for the 3D printing. They consider the following steps: analysis of planimetric configurations, analysis of the geometric components, 3D modelling on an ideal shape, evaluation of the deviations between the ideal shape and the 3D point cloud data, unfolding conic surfaces for the purpose of high quality ortho generation.

In some sense, the methodology that we propose can be viewed as a synthesis of the above presented approaches in what concerns the use of point clouds as the primary source of data for the geometrical interpretation and analysis of the vault constructive system, and as a benchmark for the qualitative and quantitative evaluation of those interpretations. The core of the methodology, corresponding to its second phase, is related to geometric analysis and modelling, and includes a few key steps that need to be highlighted in advance for the sake of clarity of reading: a) definition of the planimetric bounding geometry, b) sectioning the point clouds through key plans to extract the main profiles, c) analysis of those profiles in terms of geometric configuration, d) in the more complex cases, elaborating on the geometric nature of the geometric elements present, e) developing an idealized 3D model to be compared with the point cloud data.

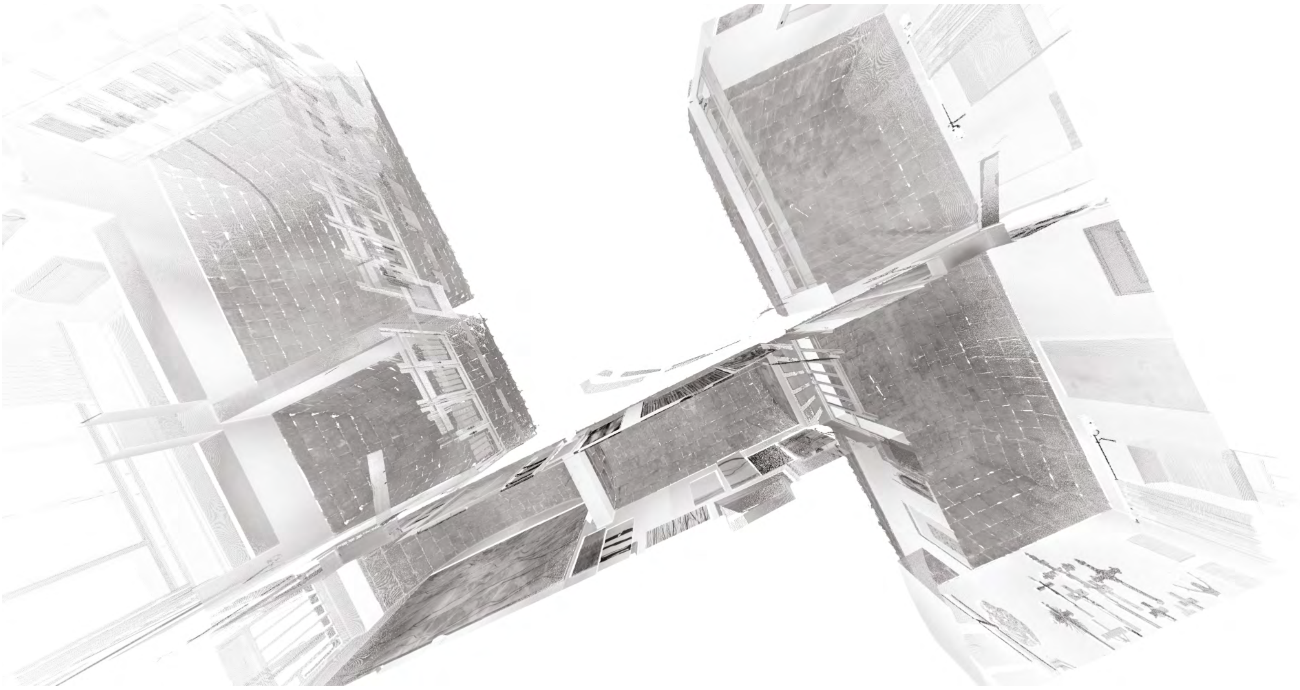


Fig. 1 Point cloud data: Inferior view of the vaults covering the spaces of a vernacular house in Alentejo. Source: author, 2024.

The case studies

To apply the methodology, we have chosen three situations of vault construction: 1) one vault of vernacular architecture in Alentejo, 2) one vault of the house of Valflores, from 16th century, in Loures, and 3) the main vault of the Arco da Rua Augusta, in Lisbon. The purpose is not to provide a historical or typological study of vaults but rather to establish a possible path for the analysis of that type of constructive element which, in turn, can bring some light on its heritage value. The reason to choose these examples is because they represent diverse approaches to conception and construction, where it is expected different levels of sophistication in the use of geometry as generative principle.

A vault from a vernacular house in Alentejo

Alentejo traditional brick vaults are a constructive system commonly used to cover the spaces of a vernacular house [Fig. 1]. They are built without any type of support and there is no prior drawn plan for their execution. Bricks are laid out one after the other, in rows, and the progress of the construction relies on empirical

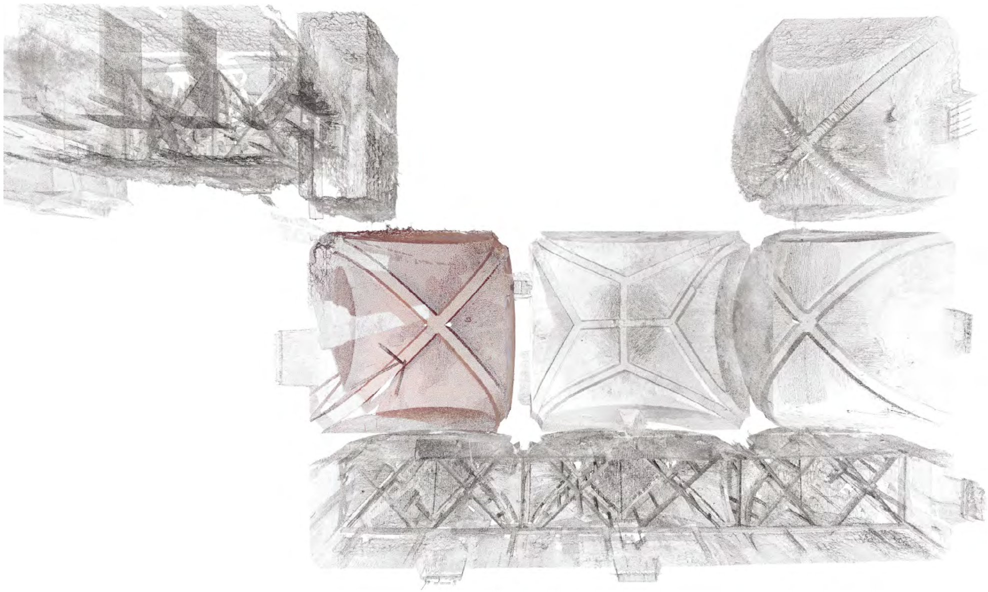


Fig. 2 Point cloud data: inferior view of the vaults covering the lower spaces of the house of Valflores. Source: author, 2024.

geometry and drying time of the mortars used. Usually, the thickness of the vault is the same as the smallest length of the bricks used, around three or four centimetres. For the application of our documentary and analytic methodology, we selected one of such vaults, commonly known as cloister vault.

A 16th century brick vault in a renaissance building in Loures

Other than the brick vaults present in vernacular architecture, one can also find this type of constructive system in more erudite buildings. The other example that we bring to our study is a vault found in the house of Valflores, a 16th century exemplar of civil architecture from renaissance in Portugal, located in Loures nearby Lisbon. This house was built by Jorge de Barros, the representative of the Portuguese king in Flanders, and one of the richest men in Portugal (Mateus 2020). In this case, the vaults are the constructive system that enables the construction of the main floor of the house [Fig. 2]. They cover the ground rooms used as warehouse. Unlike the previous type of vaults, these have a bigger thickness that corresponds to the second largest dimension of the brick, around 12cm, not considering the loads on the extrados. And they are built using brick arches as support. The data used came from the surveys done in 2016 in the context of the preparation of basis for the restoration works.

The main vault of Arco da Rua Augusta

After the earthquake of 1755, the reconstruction plan for the lower city included the construction of a large square framed by a triumphal arch aligned with Rua Augusta. The construction unfolded in two phases. In the first phase, following the building of the structures flanking the Praça do Comércio to the north, the structure was erected up to the cornice. Then, the completion was waited for several years, finally finishing in the 1870s. The complete structure is a true showroom of solutions of stone stereotomy. The constructed project is authored by the architect Veríssimo José da Costa, and the decorative program is divided between Calmels, at the top of the arch, with sculptures representing Glory crowning Genius and Valor, and Vítor Bastos, with sculptures of Nuno Álvares Pereira, Viriato, Vasco da Gama, and the Marquis of Pombal (Silva 2012). Also by this author are the statues representing the rivers Tejo and Douro. Little is known about the project's author. He was appointed as a second-class artist associated with the Civil Architecture class at the Academy of Fine Arts in Lisbon,[1] is believed to have been connected to the foundation of the Association of Portuguese Civil Architects,[2] founded in 1863, and may have served as a second lieutenant in the Portuguese navy. Thus, he is a virtually unknown figure, but given what we are about to unveil, he deserves to be highlighted and studied.

The Arco da Rua Augusta **[Fig. 3]** embodies a hybrid program of Architecture and Sculpture. It is the element that marks, to the north, the symmetry of the Praça do Comércio and serves as a backdrop for the equestrian statue of the King D. José. This building, doubly pierced by the axis of Rua Augusta and the axis of the arcade of the buildings to the north of Praça do Comércio, presents itself as a triumphal arch. At the pedestrian level, its main vault is visible, above which is the clock room. This room, also vaulted, is below the terrace roof, from which one can contemplate the entire historic Lisbon. Access to the clock room is via a spiral staircase located in one of the four corners that support the structure.

Although is a relatively recent structure, just over a century and a half old, the whereabouts of the original project drawings are unknown. The only existing documentation prior to this study consisted of a set of survey drawings made in the 1960s, found in the archive of the former Direcção Geral dos Edifícios e Monumentos Nacionais [General Directorate of National Buildings and Monuments], and a topographical survey carried out by the municipality of Lisbon. These two records did not provide any detailed information about the stone cutting layout. Under the pretext of developing a conservation project for the facade surfaces of the buildings in Praça do Comércio, a 3D laser scanning survey of the Arco da Rua Augusta was done in 2010. In addition to creating the base documentation for the

[1] ANTT, Registo Geral de Mercês, Mercês de D. Maria II, Lv. 9, fol. 11.

[2] Museu Arqueológico do Carmo, website: https://museuarqueologicodocarmo.pt/historia.html, cons. 04.02.2015.

Fig. 3 Point cloud data: Inferior view of the main vault of Arco da Rua Augusta. Source: author, 2024.

restoration project developed by Atelier 15, led by architects Alexandre Alves Costa and Sérgio Fernández, it was possible to conduct an initial study of the generative geometry of the main vault (Mateus et al. 2012), which is now deepened through a simulation of what could be the stereotomy of that structure, inferred from the visible surfaces recorded in the survey.

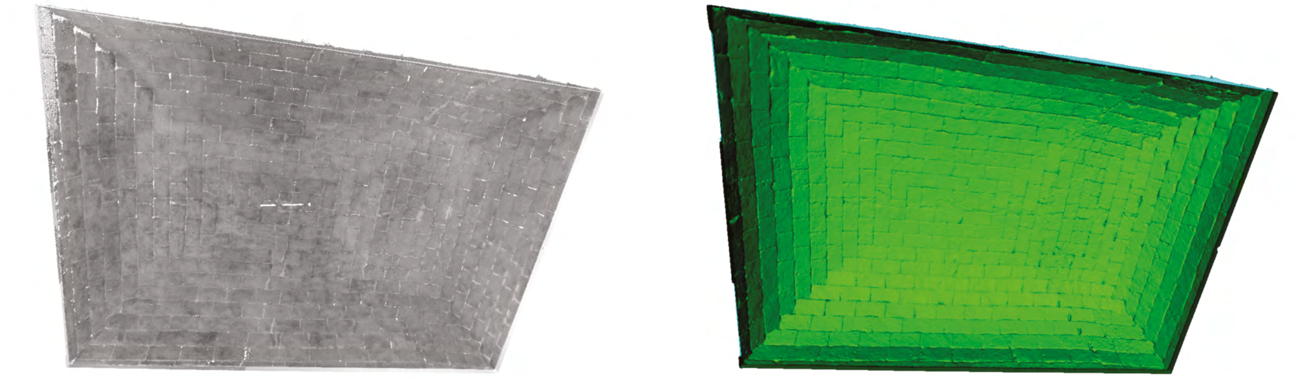
Recovering the geometry from empirical making

In this section it is presented the first case study about a vernacular brick vault.

Laser scanning survey and mesh modelling

Laser Scanning provides the raw data from which all kinds of geometrical analysis and modelling can be performed. In Fig. 4, in the left, we have an image of a point cloud, and in the right, we have its conversion to a triangulated model, commonly known as mesh.

Fig. 4 Left: point cloud. Right: mesh model.



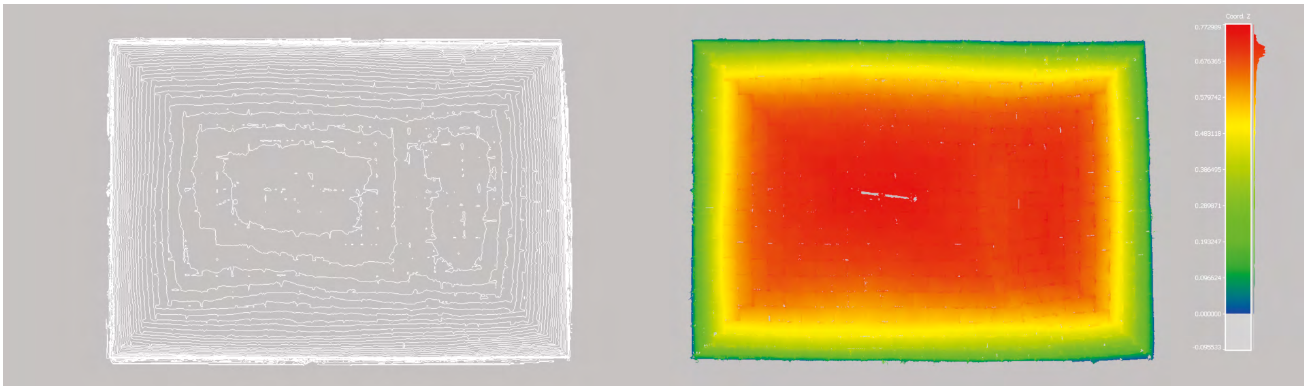


Fig. 5 Left: contours. Right: height map.
Source: author, 2024.

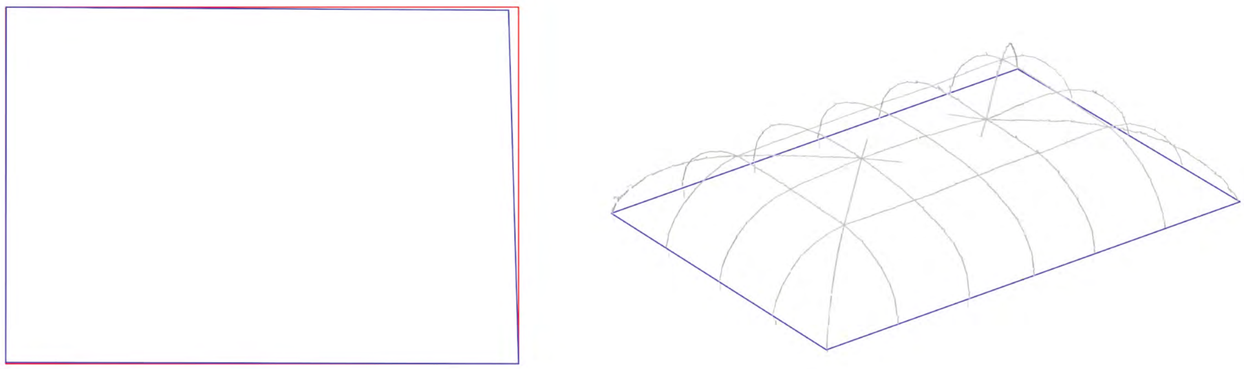
Both representations can be very detailed and can depict the deformations of the real objects with high precision and accuracy. They constitute the primary deliverables of a laser scanning survey. From a conservation and restoration perspective these are very important because they can document the actual state of the geometry and provide valuable information about the conservation and structural condition. So, whenever we want to extract data that represents the existing objects accurately, these types of models are the ones to be used. For example, in the images of **Fig. 5**, we can understand the deformation of the vault in two different ways. In the left side we can understand it by looking at the contours every 2.5cm, and in the right side the deformation becomes apparent when we associate each point with a colour that represents its height. These are two typical representations that we can use to show and quantify the deformation of an existing geometry.

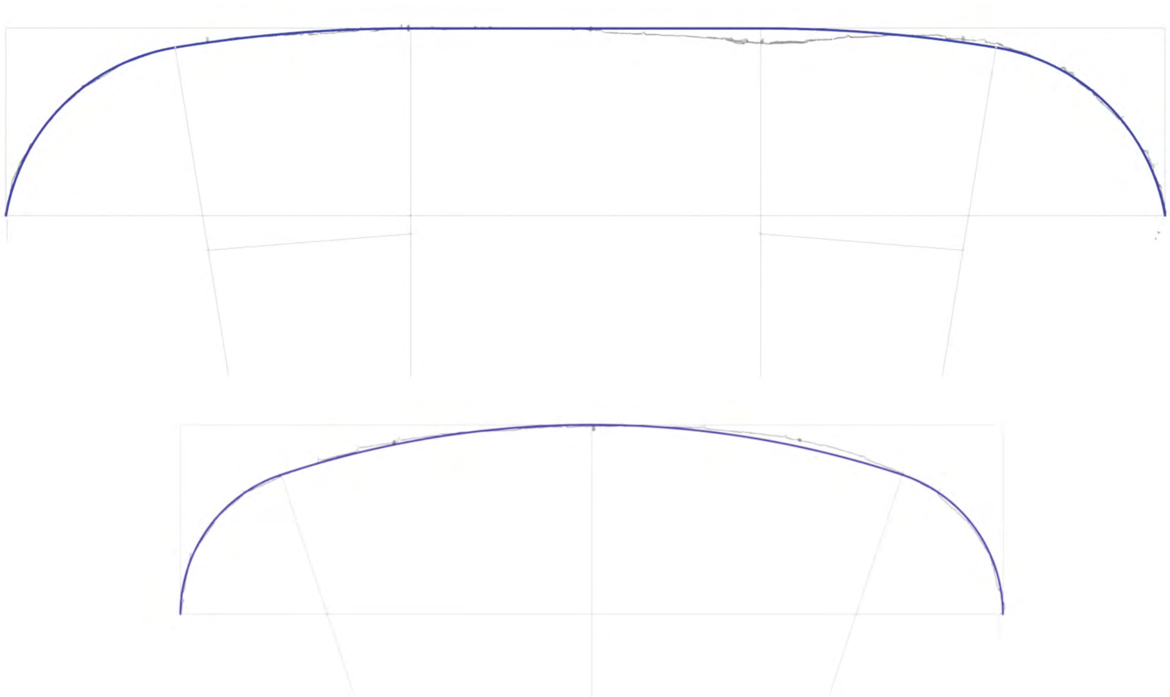
Geometric analysis and modelling of the vault

Although the previous representations depict the actual geometry very accurately, one may be interested in understanding the geometry from a generative point of view. This means to understand what are the primitive geometric interpretations that we can associate with the built reality. Is this geometry based on rectangles, circular

Fig. 6 Left: Difference between bounding rectangle and bounding quadrilateral. Right: Main sections extracted from the point cloud. Source: author, 2024.

Fig. 7 Top: Longitudinal arc. Bottom: Transversal arc. Source: author, 2024.





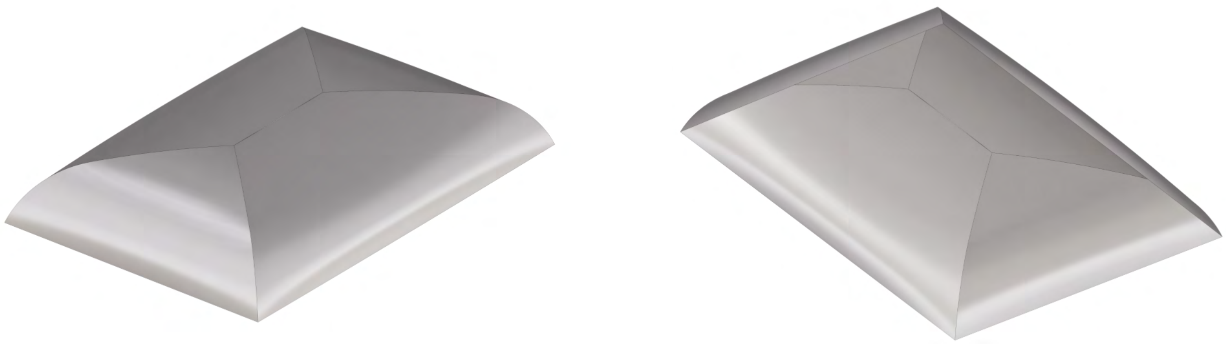


Fig. 8 Final geometrical model.
Left: view from top. Right: view from bottom.
Source: author, 2024.

arcs, or in more complex geometries? The response to these questions usually arises with the need to model the geometry in the context of simplified representations as it can occur when developing other types of models. And it is a trade-off between considering a very idealized geometry and a geometry that fits the data up to a given threshold. For example, in this case, the most simplified geometric representation would be inscribed in rectangle, but that may not acceptable because such assumption may imply a significant deviation between the representation and the real data. This is always an issue when modelling historic constructions because they almost always present a lack of orthogonality. One approach can be to introduce small transformations to the idealized geometries to make the models fit better the existing data without introducing unnecessary complexity. In this example, in the left side of **Fig. 6**, we can appreciate the difference between a bounding rectangle (in red) and a bounding quadrilateral (in blue). The rectangle measures 4.66m x 3.24m and in the most discrepant point the difference between the rectangle and the quadrilateral reaches almost 10cm. Of course, neither the rectangle nor the quadrilateral represent exactly the bounding geometry of the vault but the second is a better approximation and does not introduce much complexity. Therefore it was adopted in this case.

Then, starting with the quadrilateral we can extract a set of key sections to be used to understand the generative geometry of the vault **[Fig. 6, right]**. Now, we need to check what is the simplest geometry that can represent those sections, and we need to decide the level to which it should fit the existing data. In **Fig. 7** we present representations for the longitudinal section of the vault (top) and transversal section of the vault.

We observe that, for each half arc, excluding deformations, the geometry can be captured using two tangent circular arcs. In the transversal section it was assumed a symmetry. It would also be reasonable to search for a better approximation to the right side of the arc. The same exercise can be done for the other sections shown in **Fig. 6**.

From these geometrized sections we can obtain a simplified, yet precise, model of the vault visible surface **[Fig. 8]**.

Deviations between the surveyed data and the idealised geometric model

One way to verify the level of compliance of the geometric model with the surveyed data is to measure the distances between both. In this case these differences will reflect both the geometrical simplification introduced and the fact that some deformations were not considered **[Fig. 9]**.

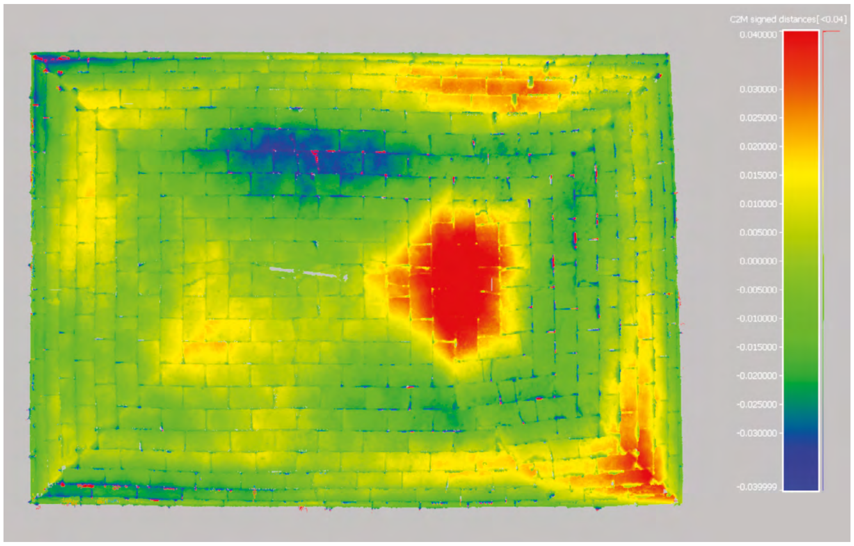


Fig. 9 Distance analysis between the modelled surface and the surveyed point cloud.
Source: author, 2024.

In Table 1 we can see that, despite the modelled surface did not consider the structural deformations of the vault, about 50% of the points in the point cloud are within a range of 1cm to the model. This means that the approximation to this representation suits most of architectural representation purposes. On the other hand, the analysis performed makes the structural deformation of the vault very evident and quantifiable.

Point cloud	Number of points	Percentage of points
Vault base point cloud	5.280.730	100%
Segmented point cloud (± 1.5 cm)	3.824.232	72%
Segmented point cloud (± 1.0 cm)	2.803.029	53%
Segmented point cloud (± 0.5 cm)	1.510.896	29%

Table 1. Distance analysis between the geometrical model and surveyed point cloud.

3D printing as way to present a constructive detail

Finally, in this example, 3D printing was used to put in evidence some features of the constructive system that are lost in the above simplified representations. In Fig. 10 (right) we can see that the rows of the bricks do not align in the edge of

Fig. 10 Left: 3D printing of the vault. Right: 3D printing of a detail. Source: author, 2024.

the vault. This type of model can have a pedagogical role in the field of construction history. It can be used to teach about how the bricks are placed in the construction phase to ensure an equilibrium of the structure. The haptic nature of 3D printed models makes them richer tool in explaining certain geometrical configurations than drawings. For example, in this case, the way the two reference surfaces meet in the corners, without a defined curved edge, would be very difficult to convey and understand through drawings. On the other hand, the model, although being a compact one, perfectly allows one to understand how much each row of bricks must be displaced aside to enable the variation of slope for the orthogonal row.

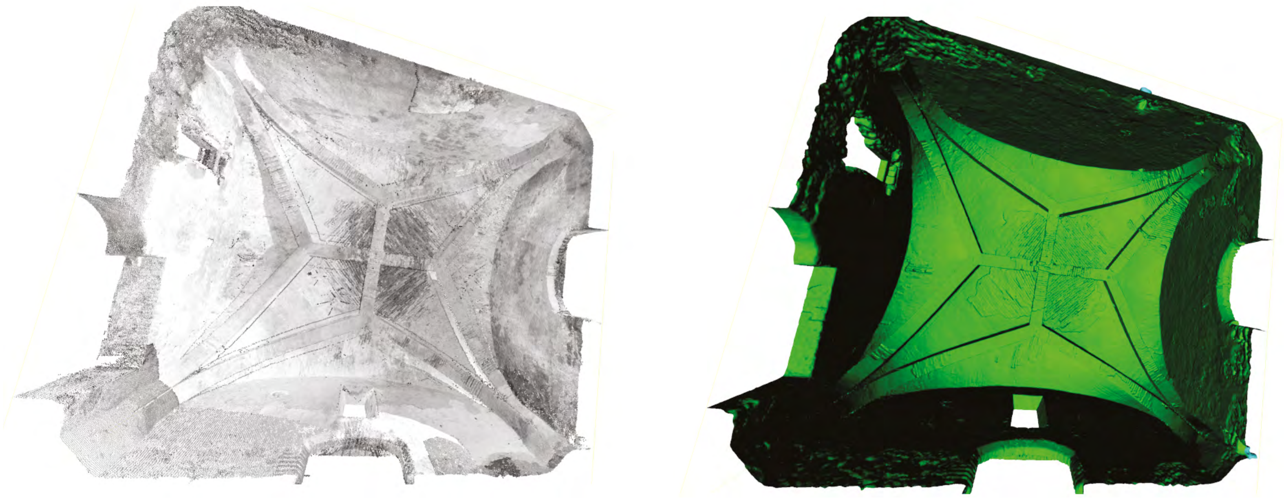
Brick vaults in a 16th century building

In this section, the second case study is presented.

Laser scanning survey and mesh modelling

Like in the previous example, the point cloud and its corresponding mesh provide a detailed description of the visible surfaces of the vault. In this case, the state of conservation of the structure becomes apparent as we can observe the lack of coating mortars, particularly in the centre of the vault. [Fig. 11]
At the naked eye, this vault appears to be perfectly symmetric. However, if we segment the height map as we can see in Fig. 12, then we can perceive that there is no perfect symmetry.

Fig. 11 Left: point cloud. Right: mesh model.



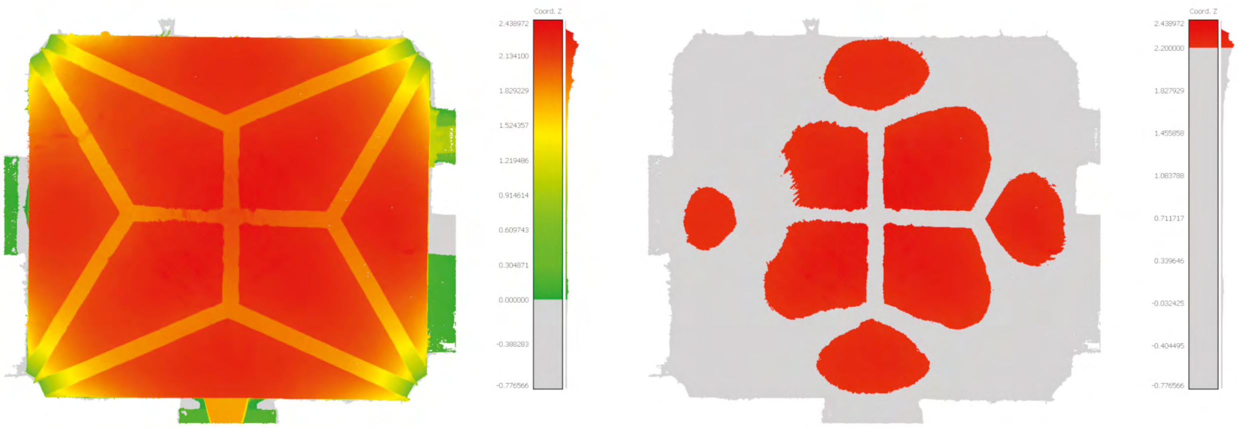


Fig. 12 Left: height map. Right: upper part of the height map. Source: author, 2024.

Geometric analysis and modelling of the vault

Like in the previous example, we propose that the geometric analysis starts with extracting the polygon corresponding to the room where the vault is located. If the polygon is similar to a rectangle within a few centimetres, one can consider the rectangle as good representation of the room, otherwise, the polygon configuration should prevail. There is no universal rule to justify this decision. What should be considered is the purpose for which the analysis is being conducted. From a modelling point of view, both options should be possible. What we mean is that a simpler representation should not be justified just because there is a limitation of the modelling tool. In our example, although the symmetry of the vault is not perfect, the room can be well represented by a rectangle with 6.18m x 5.56m.

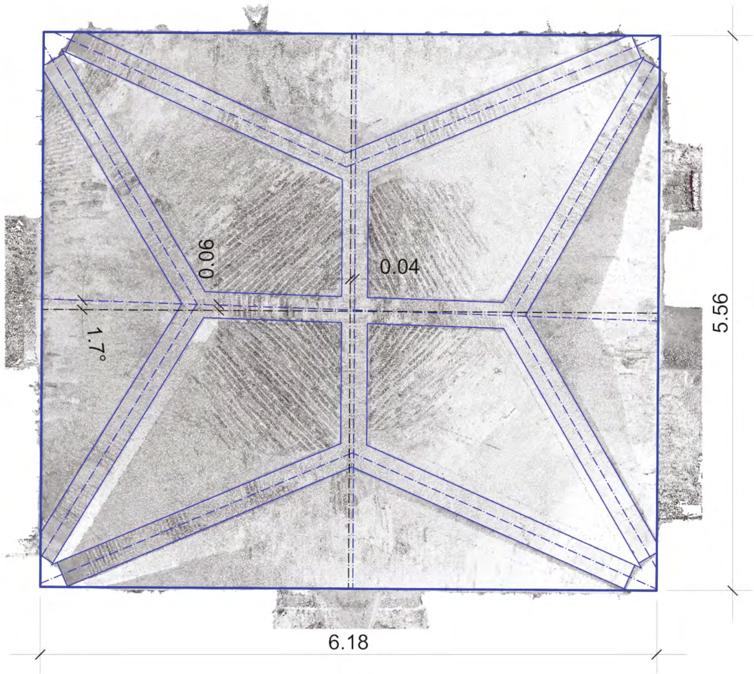


Fig. 13 Room axes (in black) and ribs axes (in blue). Source: author, 2024.

The deviations from this geometry are due to the irregularities of the walls and do not exceed 3cm. Although the room can be considered symmetrical, the main ribs axes (blue) have some deviations from the room axes (black), both in location as in orientation, as it can be seen in **Fig. 13**.

Again, one should ask if it is admissible to model the vault fitting the room as if it was symmetrical or should consider these deviations. As previously mentioned, there is no universal rule to make this choice. Our option was to consider these deviations. However, we opted to model the surfaces of the ribs as if there was not any lack of coating mortars. The next step in the modelling process is to see what geometry best represents the arcs of the ribs. To do so, we sectioned the point cloud through vertical planes containing the axes. Then we fitted primitive geometries. This was a progressive process. This means that, within the desired level of approximation, if a simpler geometry can be used, that should be preferred to a more complex one. For example, if possible, a circular arc is preferable to an elliptical arc. The exception to this rule exists if we have reasons to believe that, from a conceptual and generative point of view, the more complicated geometry represents better the existing configuration. If we conclude that there is enough repetition or that all geometry is from the same family, we can use parametric modelling to facilitate the process. The two central arcs can be correctly represented

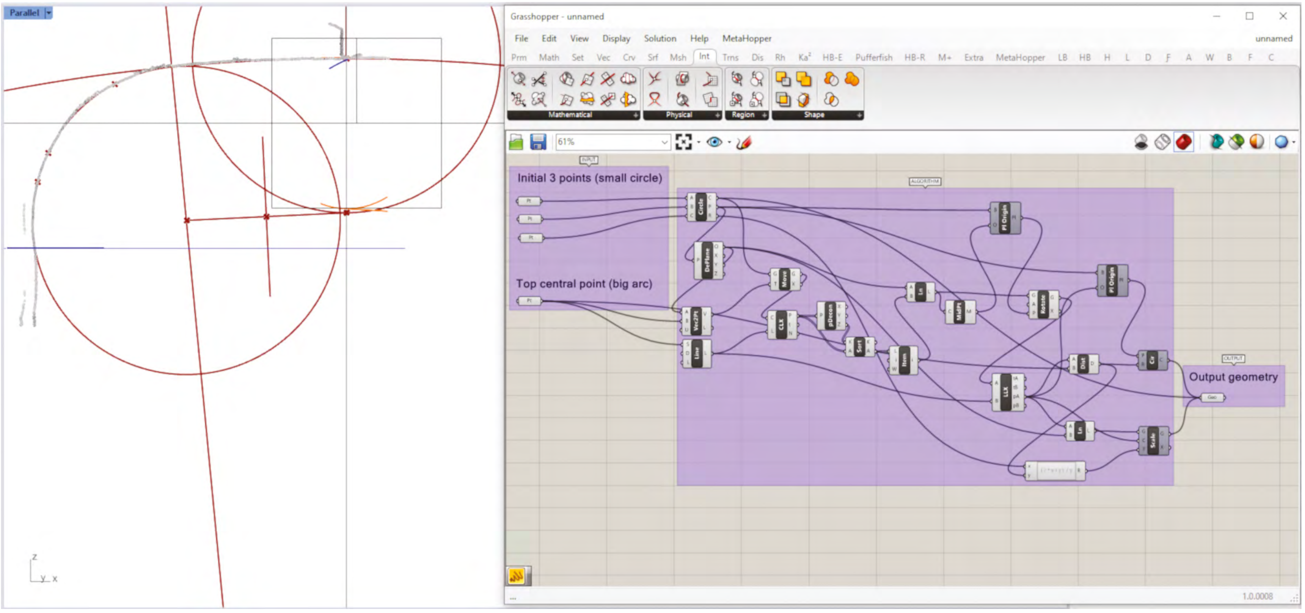


Fig. 14 Parametric model of the corner ribs arcs. Source: author, 2024.

by circular arcs. The other eight arcs that start in the corners of the room can be correctly represented by two circular tangent arcs. For these we used parametric modelling [Fig. 14].

The advantage of parametric modelling is the possibility of repetition within the family of geometries to be modelled. Small changes in the input parameters become easily reflected in the output models. And that comes with considerable savings in modelling time.

Deviations between the surveyed data and the idealised geometric model

Since we modelled the ribs and the spans of the vault as if there was not any lack of coating mortars, the deviations obtained with this analysis reflect more the quantity of missing mortars than the geometric quality of the model. In Fig. 15 that effect is clearly seen in the central area of the vault, where the texture caused by the bricks is also visible. If needed, the difference between the point cloud and the model could be used to estimate the volume of missing mortars. This could be relevant in case of restoration works.

In the table below, we can see that 61% of the points in point cloud are within a range of 1cm to the geometric model.

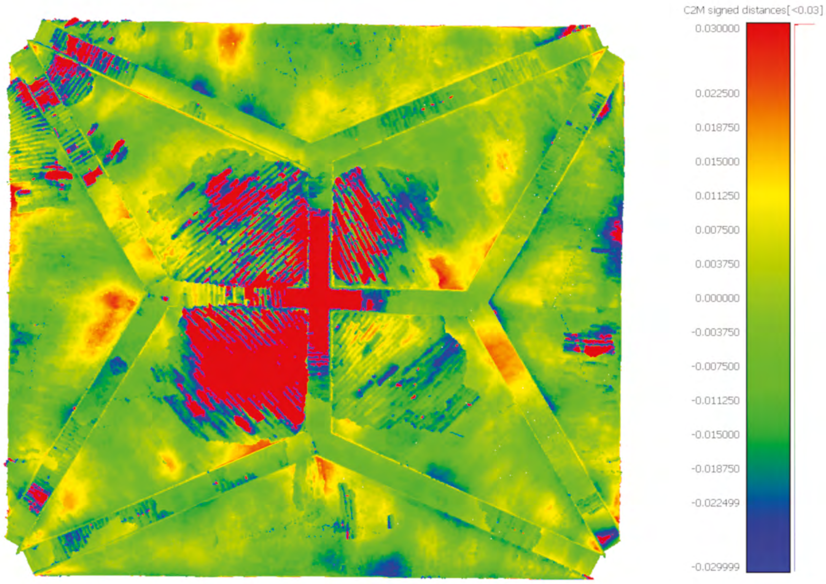


Fig. 15 Distance analysis between the modelled surface and the surveyed point cloud. Source: author, 2024.

Point cloud	Number of points	Percentage of points
Vault base point cloud	480.008	100%
Segmented point cloud (± 1.5 cm)	373.384	78%
Segmented point cloud (± 1.0 cm)	294.698	61%
Segmented point cloud (± 0.5 cm)	161.827	34%

Table 2. Distance analysis between the geometrical model and surveyed point cloud.

3D printing to convey a volumetric idea

3D printing can be used for many purposes like showing a general view of the geometry, illustrating a constructive principle, or put in evidence a particular part of the object. With this example, our intention was to show a global view of vault [Fig. 16]. It can be argued that a simple visualisation of a 3D model can play a similar role. And that is true to some extent. However, there are features that are easier to understand by handling a physical model. For example, subtle discontinuities in surfaces, like the ones that happen in this case in the transitions between ribs, are easier to see in a scaled physical model than in a digital model. Additionally, we can also think about these models as medium to make heritage accessible to impaired people or other audiences, or even construction sites, without being dependent on digital representations.

Fig. 16 3D printing of the ribs and spans of the vault. Source: author, 2024.

Geometry as the base for stereotomy

Finally, in this section it is presented the last case study.

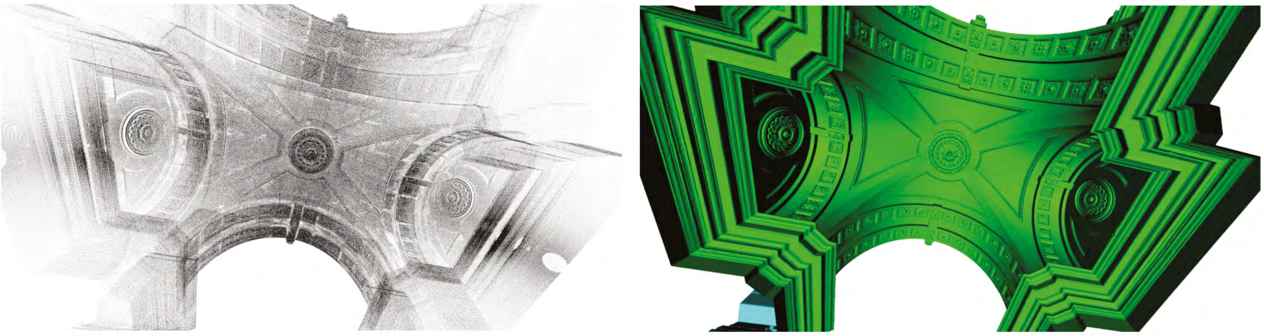
Laser scanning survey and mesh modelling

The laser scanning survey initially conducted in the context of the preparation of conservation works included the recording of the building's exterior and interior surfaces in the form of point clouds **[Fig. 17 left]** and allowed for its graphical representation through floor plans, sections, and elevations, enriched by orthoimages. These elements proved to be fundamental for the analysis presented. From a 3D modelling point of view, the point cloud is a base to produce several types of models. These can depict all the details **[Fig. 17 right]** or can be more idealized. If the purpose is to get other types of models, like CAD or BIM, it is mandatory a trade-off between full accuracy and efficiency of the model.

Geometric analysis and modelling of the vault

The layout defined by the relationship between floor plan, section, and elevation in some way reflects the operational setup of the 19th century architect. In this

Fig. 17 Left: point cloud. Right: mesh model.



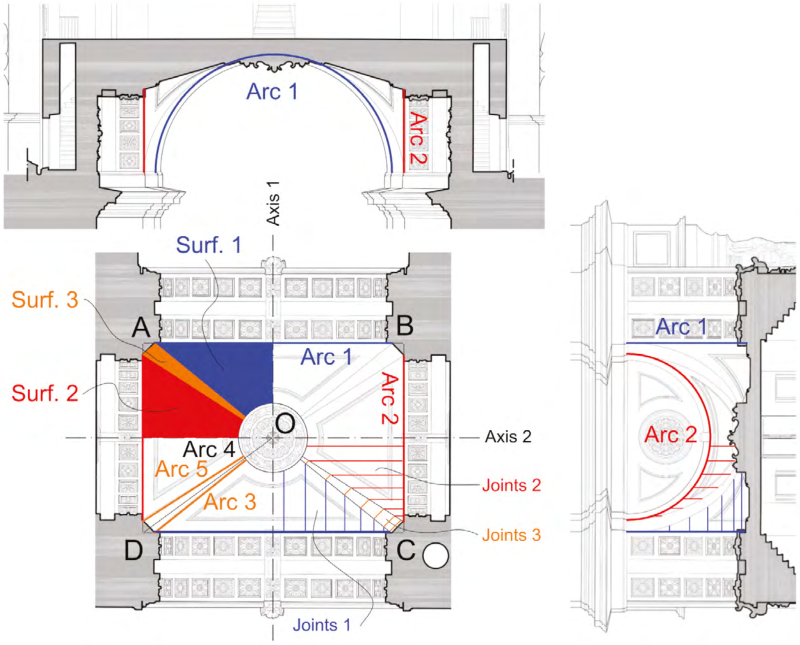


Fig. 18 Recapturing the operative layout of the architect through plan and sections. Source: author, 2024.

sense, it is more significant for understanding the generation of form than an exclusive reading of the three-dimensional data resulting from the survey.
By doing this kind of reverse engineering of the design project, a set of geometric relationships, that would otherwise remain hidden, is revealed **[Fig. 18]**.
Through **Fig. 18**, it becomes evident that the geometric structure of the vault has two axes of symmetry: one transversal (axis 1), aligned with Rua Augusta, and another longitudinal (axis 2), aligned with the arcades. The area of the vault corresponds to the rectangle [ABCD], with centre O. Axis 1 contains the centres of arcs of type 1, and axis 2 contains the centres of arcs of type 2. These arcs correspond to semicircles contained in vertical planes, with the radius of arc type 1 being greater than the radius of arc type 2. It is then observed that arcs of type 1 give rise to surfaces of type 1 through extrusion. Thus, surfaces of type 1 are cylindrical, which is also confirmed by the drawing of joints 1, parallel to axis 1. The cylindrical surfaces, in turn, are bounded by vertical planes according to arches of type 3. Consequently, arcs of type 3 are elliptical.

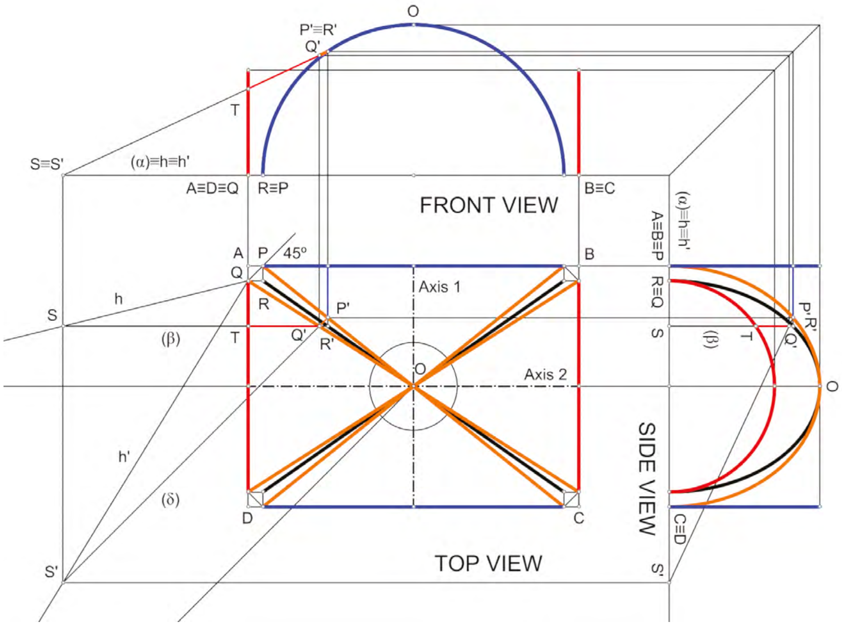


Fig. 19 Geometric analysis of surfaces 2 and 3. Source: author, 2024.

Next, we postulate the consideration of arcs of type 4, which also result from the intersections of the cylindrical surfaces with diagonal vertical planes of the rectangle [ABCD]. In a manner like what occurs with arcs of type 3, the arcs of type 4, which do not correspond to any edges of the vault, are also elliptical. These arcs of type 4, together with arcs of type 2, support the generation of surfaces of type 2. In these surfaces, it is observed in the ceiling plan and in the side view that joints 2 are parallel to the vertical plane of axis 2. Considering this observation, at first glance, we are led to conclude that surfaces of type 2 are cylindroids, that is, ruled surfaces, with a directing plane, whose generatrices rest on two curved directrices.

Subsequently, the surfaces of type 2 are bounded by the arcs of type 2 and type 5, whose geometric nature remains to be clarified, contained in vertical planes. Finally, the arcs of type 3 and 5 bound the surfaces of type 3, which, due to having the joints 3 parallel in the plan view, are also supposed be cylindroids.

However, deepening the geometric analysis of surfaces of type 2 and 3, it is verified that they are not cylindroids, as we initially supposed, but rather conoids **[Fig. 19]**.

Let us begin with surfaces of type 2. A surface of this type is guided by one arc of type 2 and one arc of type 4, and a directrix plane. The directrix plane is vertical with an orientation β parallel to axis 2, which means that all the generatrices of this surface, [R′T], will be parallel to axis 2 in the plan view. Since arc 2 and arc 4 are conic curves, specifically a circle and an ellipse, in the side view they are related by an affinity of axis h. That is, for any generatrix [R'T] of this surface, the ratio R'T/TS is constant, as manifested in the side view, in the plan view, and also in the front view. This means that, in the top view, the lines representing the arcs 4 and 2 are also related through an affinity of axis h. In fact, line h is the trace of surface 2 in plane α, which is the horizontal plane above which the vault rises. Therefore, it follows that although we started with two curved directrices, surface 2 also contains a straight one, line h, which makes it a conoid.

Regarding surfaces of type 3, it is noted that they result from the chamfering the corners of rectangle [ABCD] at 45º in relation to its sides. That is, surfaces of type 3 connect cylindrical surfaces of type 1 to conoid surfaces of type 2. These surfaces have a vertical directrix plane with orientation δ at 45º in relation to axes 1 and 2. The directrices of these surfaces are arcs of type 3, which are elliptical, and arcs of type 5, whose nature is unspecified. For example, the arc 5, contained in the vertical plane passing through points Q and O, results from the intersection of this plane with the corresponding conoid surface of type 2. For a generatrix [TQ'] of the conoid surface of type 2, there corresponds a generatrix [P'Q'] of a surface of type 3. It is verified that these two generatrices and the generatrix of the cylindrical surface of type 1, which passes through point P', are coplanar. For this reason, the trace of the generatrix [P'Q'] in plane α, line S.S', has the direction of axis 1. Now, note that all the triangles of type [Q'SS'], in the plan view, must all be homothetic with respect to point Q. Hence, all points of type S' must be located on a line h' passing through point Q. This line h' is the trace of the surface of type 3 in plane α. Thus, by considerations analogous to those made above, surfaces of type 3 are also conoids.

If we proceed with a first modelling attempt of the vault based on this analysis, we will see what can be observed in **Fig. 20**.

In this model, the appearance of two vertices, X and Y, is observed, which are the ends of the junction edges of surfaces of type 2. However, these edges are incompatible with the data from the survey. This means that there should exist new surfaces ensuring a smooth transition between the two surfaces of type 2. Geometrically, the connection between the two surfaces of type 2 should be made through tangency with another surface. Our hypothesis consists in assuming that this new surface is also a conoid.

Since surfaces 2 are warped, the transition to other surfaces, which is assumed to be along a straight generatrix, must consider an auxiliary quadric, a hyperbolic paraboloid, or a scalene warped hyperboloid, simultaneously tangent with the surface of type 2 and with the new surface, which is also ruled, along the common straight generatrix. The new surface is also a conoid with directrix plane β. Furthermore, this new surface must be symmetrical with respect to axis 2. We will

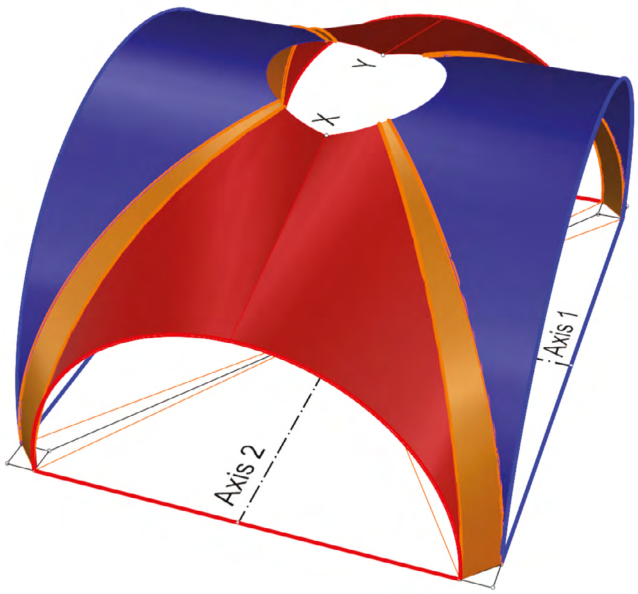


Fig. 20 First model of the vault where it is perceived an edge between surfaces 2. Source: author, 2024.

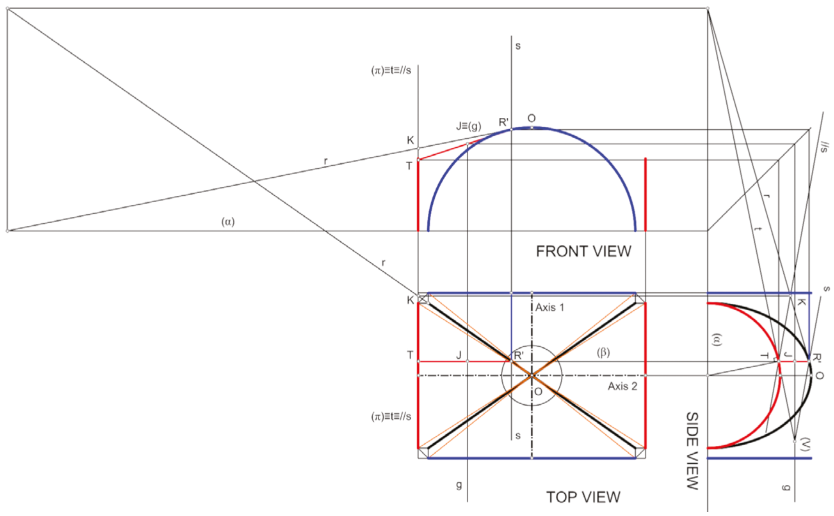


Fig. 21 Smoothing the edge between surfaces 2 through the tangency with another surface. Source: author, 2024.

now return to the representation of Fig. 19 to resolve the tangency we just mentioned. The drawing is what is shown in Fig. 21.

The first step is to select the generatrix of tangency, in this case, line T.R′, which is done analysing the surveyed data. This line intersects arcs 2 and 4 at points T and R′, respectively. The tangent plane to conoid 2 at point T is defined by line t, tangent to arc 2 at point T, and by line T.R′. The tangent plane to the conoid at R′ is defined by line r, tangent to arc 4 at point R′, and by line T.R′. These two planes will be tangent to the conoid surface that we intend to define, as well as

Fig. 22 Final geometrical model.
Left: view from top. Right: view from bottom.
Source: author, 2024.

to the auxiliary quadric that needs to be considered, in this case, a hyperbolic paraboloid with directrix planes β and π. Orientation π is chosen because this is the only orientation that ensures a symmetry regarding the axis 2.

The tangent plane to the conoid 2 at point R′ intersects plane π along line //s, defined by points K and T. This line gives us the direction of generatrix s of the hyperbolic paraboloid passing through R′, the straight line s. Since there is a family of generatrices of the hyperbolic paraboloid parallel to β, this gives rise to the point of divergence V, in the side view, through which the generatrix g of the hyperbolic paraboloid can be drawn, horizontal and parallel to π. This generatrix g is also be common to the transitional conoid between the two surfaces of type 2. Thus, the transitional conoid between surfaces 2 will have β as the directrix plane and arcs 2 and line g as its directrices.

The modelling of the vault, taking into account this new surface, and refined by considering surfaces parallel to those studied to accommodate the profiles and the trimming in the stone cutting, can be seen in Fig. 22.

Deviations between the surveyed data and the idealised geometric model

The final test of the hypothesis regarding the generative geometry of the vault was conducted by analysing the distances between the point cloud from the survey and the model resulting from the previous stage [Fig. 23].

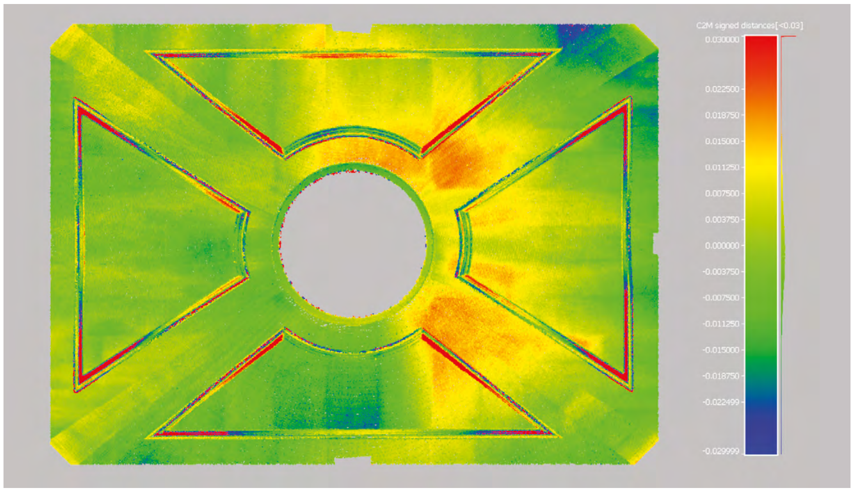


Fig. 23 Distance analysis between the modelled surface and the surveyed point cloud.
Source: author, 2024.

Table 3 shows that there is a high degree of congruence between the produced model and the survey data. It is observed that about 71% of the points in the point cloud are within a distance less than 1 cm from the produced model. These results are even more significant regarding the quality of the construction when considering that the adjustment was made between a geometrically idealized and perfectly symmetrical model and real data, which naturally reflect the asymmetries and imperfections inherent to the construction process, as well as slight deformations resulting from the passage of time.

Point cloud	Number of points	Percentage of points
Vault base point cloud	719.529	100%
Segmented point cloud (± 1.5 cm)	624.552	87%
Segmented point cloud (± 1.0 cm)	509.077	71%
Segmented point cloud (± 0.5 cm)	312.408	43%

Table 3. Distance analysis between the geometrical model and surveyed point cloud.

Modelling and 3D printing of a stereotomic hypothesis

Based on the model of the visible surfaces of the vault, and taking into account the joints between the voussoirs visible in the point cloud from the survey, a hypoth-

Fig. 24 Stereotomic hypothesis.
Left: view from top. Right: view from bottom.
Source: author, 2024.

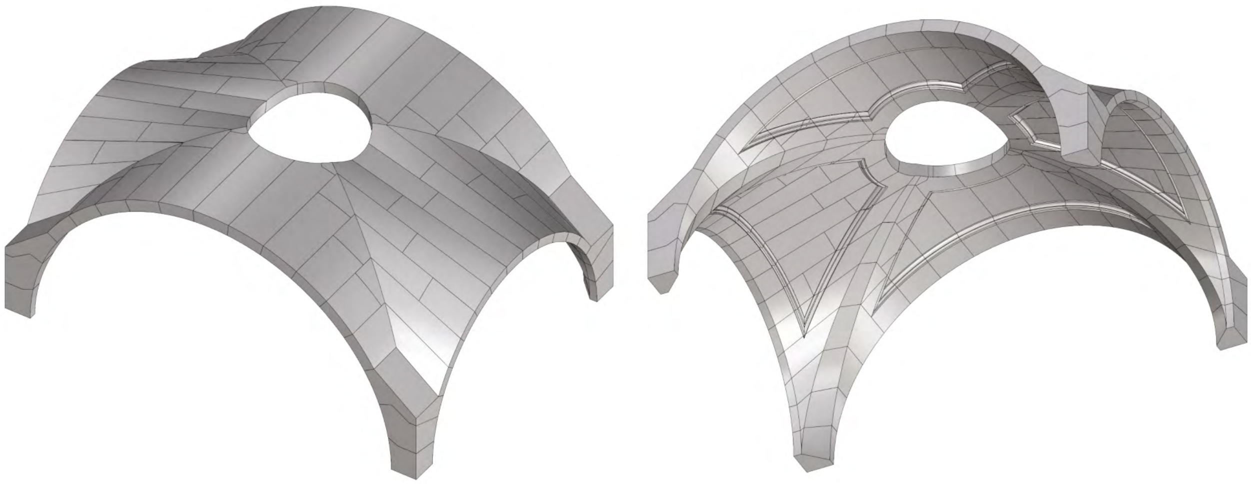


Fig. 25 Process of assembly of the voussoirs after 3D printing. Source: author, 2024.

esis of stereotomy was modelled, estimating the thicknesses and the geometry of the contact surfaces between the stone blocks, as can be seen in Fig. 24.
Regarding the thickness, an average value of 30 cm was considered, from which the lower trimming of 14 cm was subtracted. As for the non-visible joints between stones, it was assumed that they are flat and convergent along axes 1 and 2. A total of 70 voussoirs were modelled, including four that form the closure of the vault (not represented in Fig. 24). Assuming a specific weight of 2500 kg/m³ for the limestone from which the vault is constructed, the weight of the voussoirs varies between half a ton and two tons. Each of the closure voussoirs of the vault weights approximately three and a half tons.
Finally, each of the modelled voussoirs was 3D printed at a scale of 1/20 and assembled with the help of a cardboard structure that reproduces the boundary conditions of the constructed vault, restricting the sliding movement of the pieces. To simulate the friction between the pieces and any metallic elements that may exist, adhesive tape was used. It was observed [Fig. 25] that during the assembly process, the structure immediately began to behave like an arch and vault. Note that even before it was completely assembled, the pieces were already self-supporting. When the pieces corresponding to the closure were added to the assembly, the final model became stable and capable of withstanding some pressure.
This type of physical model can be used to gain insights about the actual behaviour of a vault structure and to simulate collapsing mechanisms.

Discussion and conclusions

In this article, we presented a methodology for the documentation and geometrical analysis of the constructive system named vault. We applied it to the study of three examples of vaults representing different types of architecture, namely a traditional brick vault in a vernacular house, a brick vault in a renaissance house, and the main vault of the Arco da Rua Augusta, a remarkable example of stone stereotomy. The methodology included the phases of surveying, geometric analysis, three-dimensional modelling, comparison of the generative geometry with the surveyed data, and 3D printing. Documentation and recording methods such as laser scanning and photogrammetry are increasingly established as the standard that ensures the precision and quality of data, enabling the development of more rigorous quantitative studies which are fundamental to inform intervention actions.
Since a vault is used to cover a given space, the geometric analysis started from a planimetric view to define the bounding polygon representing that space. In the

case studies of the brick vaults we observed that the actual bounding polygons may deviate significantly from a rectangle (vernacular vault), and, when that happened, they were chosen to represent the space being covered by the vaults. In the cases of Valflores and Arco da Rua Augusta those differences were negligible and, therefore, a rectangle was considered. Then, in the cases of the brick vaults, the inference of the generative geometry was done by slicing strategically the point cloud to retrieve the curves that structure the geometry of the vault. Whenever possible, the curves were represented as circular arcs or combinations of circular arcs. Only, if circular arcs did not represent the curves correctly, other geometries were adopted, which did not happen in these cases. For the Arco da Rua Augusta it was followed a different approach. We laid out a two-dimensional layout that related the plan and two sections, one transverse and the other longitudinal. In this way, the typical operational setup of a 19th century architect was recovered, as it is the case with Architect Verîssimo José da Costa, author of the constructed project. The geometric analysis carried out highlighted the high level of sophistication in descriptive geometry knowledge possessed by the author of the project. In particular, the use of tangent conoid surfaces in the design of the vault stood out. It is important to note that ensuring the tangency between the surfaces is not trivial and involves the use of doubly ruled quadrics as auxiliary surfaces. The knowledge necessary to pursue this task was being developed in the 19th century, enhanced by the Descriptive Geometry that Gaspard Monge published at the end of the 18th century, and which others, such as Hachette, Olivier, Poncelet, Chasles, and Plücker, further developed during the 19th century. This architect was, therefore, well-informed about these developments, as evidenced by the analysis we conducted. From this perspective, the Arco da Rua Augusta is a testimony of the scientific value in that it reflects the most advanced studies of geometric surfaces existing at the time and the results obtained with this case study constitute, themselves, a contribution to the study of stereotomy in Portugal in the 19th century, period for which there is no research developed.

Considering the comparison made between the produced models and the surveyed data, it was found that the accordance between the geometric models and the surveyed data, expressed by the distances between both, showed that the percentage of points that range within a distance less than one centimetre is 53%, 61% and 71%, for case studies 1, 2 and 3. These results should be read carefully because they have various meanings. For example, if an object is very deformed and deviates from the idealized geometry, this is expressed in the distances found. The same happens when there is lack of material. However, the case of Arco da Rua Augusta is remarkable when considering that this is the biggest of the three and it is about a century and a half old. We can say that the deformations in relation to the idealized model are minimal. This reflects the enormous quality of the construction, which would had only been possible with the careful design of the construction solution that must have included a planning for the stone cutting. The modelling of a hypothesis about stereotomy allowed us to provide some insights into the physicality of the object. By individualizing each of the voussoirs, it was possible to estimate their mass. This parameter can impact future studies aimed at exploring the logistics of a construction site of this type, as well as estimating what resources would be necessary to carry it out.

On the other hand, 3D printing allows physical simulations of the models. This is an important aspect from many perspectives. Printed models allow a direct physical interaction which can be used to enhance pedagogical strategies in educational contexts. These types of models can be used to improve accessibility, for example for people with impaired vision. In the case of Arco da Rua Augusta it was possible a scaled simulation of the assembly process of the voussoirs, thereby reinforcing the understanding of the tectonics of the solution. This physical model also allowed for the verification of the structural equilibrium logics of the vault. In addition to this aspect, the educational potential of this type of model for the study of historical constructive systems stands out.

This is a study that considers the built reality as a primary source of information and, therefore, intentionally does not rely on modelling solutions documented in treatises that often merely typify some standard solutions without exhausting the richness of the built reality.

Now, since the validity of the methodology is demonstrated, future work on this topic includes the analysis of other buildings from 18th and19th century Portuguese architecture. This should be a two-fold approach. On the one hand it must rely on visual analysis of the geometries to infer the generative principles but on the other hand it would be beneficial to introduce some automation to the repetitive parts of the process, which can, eventually be done by integration of artificial intelligence tools.

References

Agustín-Hernández, L., R. Argiolas, V. Bagnolo, and M. Sancho Mir. 2021. 'From TLS point cloud data to gemetrical genesis determination of ribbed masonry vaults.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLVI-M-1-2021, 9-16, https://doi.org/10.5194/isprs-archives-XLVI-M-1-2021-9-2021

Attenni, M., R. Bami, C. Bianchini, M. Griffo, C. Inglese, Y. Levy, D. Pritchard, and G. Villa. 2023. 'The vaulting system of the Palatine chapel: The Aachen cathedral world hertitage site documentation project.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLVIII-M-2-2023, 119-128, https://doi.org/10.5194/isprs-archives-XLVIII-M-2-2023-119-202

Banfi, F. 2019. 'HBIM generation: Extending geometric primitives and BIM modelling tools for heritage structures and complex vaulted systems.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLII-2/W15, 139-148, https://doi.org/10.5194/isprs-archives-XLII-2-W15-139-2019

Callabria, C. 2015. 'Construction and Compositions of the 16th century multi ribbed vault in Portugal. Conceptual transposition in the contemporary project.' PhD thesis. Polytechnic School of Bari.

Calvo-López, José. 2020. *Stereotomy. Stone Construction and Geometry in Western Europe. 1200-1900.* Basel: Birkhäuser.

Capone, M., and E. Lanzara, 2019. '3D data interpretation using treatises geometric rules to built coffered domes.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLII-2/W15, 231-238, https://doi.org/10.5194/isprs-archives-XLII-2-W15-231-2019

Derand, François. 1643. *L'Architecture des Voutes, ou L'art des Traits, et coupe des voutes*. Paris: Sebastien Cramoisy.

Delgado, Ricardo Jorge Conduto. 2017. 'A geometria na estereotomia da pedra na arquitectura religiosa portuguesa entre 1530 e 1580.' PhD thesis. Faculdade de Belas Artes da Universidade de Lisboa.

Escudero, Lorenzo de la Plaza, Adoración Morales Gómez, Maria Luisa Bermejo Lopez, and José Maria Martínez Murillo. 2014. *Dicionário Visual de Arquitetura*. Lisbon: Quimera.

Frézier, Amédée-François. 1739. *La theorie et la pratique de la coupe des pierres et des bois pour la construction des voutes et autres parties des Bâtimens Civils & Militaires, ou Traité de Stereotomie a l'usage de* *l'Architecture*, vol. 3. Strasbourg; Paris: Chez Jean Daniel Doulsseker le Fils; Chez L. H. Guerin l´aîné.

Genin, Soraya. 2020. 'The vaults of Arronches Nossa Senhora da Assunção and Misericórdia churches. Geometric and constructive comparison with the nave and refectory vaults of Jerónimos Monastery.' In *Thinking, Drawing, Modelling*, edited by Vera Viana, Vítor Murtinho and João Pedro Xavier, 66-75. Berlin: Springer International Publishing.

Genin, Soraya. 2014. 'Voûtes à nervures manuélines. Le caractére innovant de João de Castilho.' PhD thesis. Faculteit Ingenieurswetenschappen, KU Leuven.

Genin, Soraya, Krista De Jonge, and José Carlos Palacios Gonzalo. 2009. 'Portuguese Vaulting Systems at the Dawn of the Early Modern Period. Between Tradition and Innovation.' In *Proceedings of the Third International Congress on Construction History*, 671-678. Brandenburg: University of Technology Cottbus.

González, Filipe Duarte. 2006. *Geometrias da Arquitectura de Terra. A sustentabilidade Geométrica das Construções em Terra Crua.* Lisbon: Universidade Lusíada Editora.

Huerta, Santiago. 2007. 'Oval domes: history, geometry, and mechanics.' *Nexus Network Journal* 9 (2): 211-248.

Lanzara, E., Samper, A. and Herrera, B. 2019. 'Point cloud segmentation and filtering to verify the geometric genesis of simple and composed vaults.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLII-2/W15, 645-652, https://doi.org/10.5194/isprs-archives-XLII-2-W15-645-2019

López González, C., Spallone, R., Vitali, M. and Natta, F. 2020. 'Baroque banded vaults: surveying and modedlling. The case study of a noble palace in Turin.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLIII-B2-2020, 871-878, https://doi.org/10.5194/isprs-archives-XLIII-B2-2020-871-2020

López González, C., García-Valldecabres, J., Escudero, P. and Meseguer, L. 2023. 'Analysis of of design patterns in buildings with overlapped architectures. Case study: The cathedral of Valencia.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLVIII-M-2-2023, 977-983, https://doi.org/10.5194/isprs-archives-XLVIII-M-2-2023-977-2023

Mateus, Luís, Nelson Brito, Victor Ferreira, Margarida Barbosa, and José Aguiar. 2012. 'New tools for visual assessment of building deformations.'

In *Proceedings of 8th International Conference on Structural Analysis of Historical Constructions*, 2463-2470. Wrocław: Dolnośląskie Wydawnictwo Edukacyjne.

Mateus, L., Ferreira, V., Aguiar, J., Ferreira, J., Mendes, C. and Silva, A. 2020. 'The role of 3D documentation for restoration interventions. The case study of Valflores in Loures, Portugal.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLIV-M-1-2020, 381-388, https://doi.org/10.5194/isprs-archives-XLIV-M-1-2020-381-2020

Palacios, José Carlos. 1990. *Trazas y cortes de cantería en el renacimiento español*. Madrid: Ministerio de Cultura, Dirección General de Bellas Artes y Archivos.

Pérouse de Montclos, Jean-Marie. 2001. *L'Architecture a la française. Du milieu du XV a la fin du XVIII siècle*. Paris: Picard.

Rabasa Díaz, Enrique. 2000. *Forma y Construcción en Piedra. De la cantería medieval a la estereotomia de siglo XIX*. Madrid: AKAL.

Rabasa Díaz, Enrique. 1996. 'Arcos esviados y puentes oblicuos El pretexto de la estereotomía en el siglo XIX.' *Púb́lica* (8): 18-29.

Rinaudo, F., Scolamiero, V., Zich, U. and Pavignano, M. 2023. 'From data to tangible models: Case study of a vault in the Royal residence of Venaria Reale.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLVIII-M-2-2023, 1315–1322, https://doi.org/10.5194/isprs-archives-XLVIII-M-2-2023-1315-2023

Rodrigues, Maria João Madeira, Pedro Fialho de Sousa, and Horácio Bonifácio. 2002. *Vocabulário Técnico e Crítico de Arquitectura*. Lisbon: Quimera.

Sakarovitch, Joel. 1997. *Epures d'architecture: de la coupe de pierres à la géométrie descriptive XVI – XIX siécles*. Basel: Birkhäuser.

Scandurra, S., M. Pulcrano, V. Cirillo, M. Campi, A. di Luggo, and O. Zerlenga. 2018. 'Integrated survey procedures for the virtual reading and fruition of historical buildings.' *Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLII-2, 1037–1044, https://doi.org/10.5194/isprs-archives-XLII-2-1037-2018

Shelby, Lonnie Royce. 1969. 'Setting Out the Keystones of Pointed Arches: A Note on Medieval 'Baugeometrie.' *Technology and Culture* 10 (4): 537..

Silva, Ana Paula. 2012. 'O Arco da Rua Augusta. História, Valores e Usos.' Master's diss. Faculdade de Arquitetura da Universidade de Lisboa.

Sousa, Pedro Fialho de. 1988. 'A estereotomia da pedra: tradição, persistência e continuidade em Portugal.' PhD thesis, Faculdade de Arquitectura da Universidade Técnica de Lisboa.

Spallone, R., M. Vitali, and F. Natta. 2019. '3D modelling between ideation, geometry, and surveyed architecture: the case of the vaulted system of "Appartamento di Mezzanotte" in Palazzo Carignano.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLII-2/W15, 1119-1126, https://doi.org/10.5194/isprs-archives-XLII-2-W15-1119-2019

Stanga, C., H. Hasniková, M. Previtali, R. Brumana, A. Grimoldi, and F. Banfi. 2019. 'The assessment of the baroque vault construction technique by scan-to-BIM process: St. Bernard's chapel in the Plasy monastery.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLII-2/W15, 1127-1134, https://doi.org/10.5194/isprs-archives-XLII-2-W15-1127-2019

Vitali, M., C. López González, G. Bertola, and F. Natta.. 2021. 'Banded vaults in Turin: TLS survey, geometric interpretation, digital re-construction, between design and construction. Palazzo Capris di Ciglìè.' *Int. Arch. Photogramm. Remote Sens. Spatial Inf. Sci.*, XLVI-M-1-2021, 793-800, https://doi.org/10.5194/isprs-archives-XLVI-M-1-2021-793-2021